

Some issues concerning the proton charge radius puzzle.

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An explanation of the difference of the charge radius of the proton as determined from the Lamb shift in electronic hydrogen and from elastic electron scattering off the proton on the one side and the recent high precision determination with muonic hydrogen on the other side is presented. It is shown that the modification of the $2S_{1/2}$ and $2P_{3/2}$ wave functions by the "Uehling potential" yields a correction to the theoretical Lamb shift of $\delta(\Delta E_{\text{Lamb}}) = 0.302 \text{ meV}$ which has to be compared to $\delta(\Delta E_{\text{Lamb}}) = 0.322(46) \text{ meV}$ equivalent to the stated radius difference. The explanation is based on the realization that the bound state wave functions modified by the external "Uehling potential" have to be propagated by the vacuum polarization propagator in order to give the correct leading order Lamb shift. It is argued that a conflicting relativistic calculation neglects this propagation aspect. The explanation demonstrates that the Lamb shift is dynamically induced through the QED vacuum polarization and is not only the result of a static external "Uehling potential" probed by a test charge.

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INTRODUCTION

The seven standard deviations difference of the root-mean-square (rms) charge radius of the proton $r_p = \langle r_p^2 \rangle^{1/2}$ derived from electron scattering and from the Lamb shift of muonic hydrogen has caused a considerable worry in the physics community. Since the Lamb shift is a corner stone of the tests of QED this difference requires indeed a convincing clarification. In this letter a proposal for an explanation in the framework of standard QED is given.

The rms radius derived from the Lamb shift in muonic hydrogen is $r_p = 0.84184(67) \text{ fm}$ [1] which has to be compared to the CODATA value of $0.8768(69) \text{ fm}$ [2] meaning a five standard deviations difference. The CODATA value is derived from a measurement of the Lamb shift in electronic hydrogen [3] and from electron scattering experiments [4]. Almost at the same time as the muonic experiment a new independent determination with elastic electron scattering has been published and yielded $0.879(8) \text{ fm}$ [5] resulting in a weighted average of all "electronic experiments" of $0.878(5)$, i.e. the mentioned seven standard deviations difference.

In order to transform this radius difference into an energy deviation of the muonic Lamb shift we repeat the key formula for the Lamb shift [1]:

$$\Delta E_{\text{Lamb}}^{\text{theory}} = (209.9779(49) - 5.2262 r_p^2 / \text{fm}^2 + 0.0347 r_p^3 / \text{fm}^3) \text{ meV} \quad (1)$$

The experiment yielded $\Delta E_{\text{Lamb}}^{\text{exp.}} = 206.2949(32) \text{ meV}$ from which the radius was determined. The term with r_p^3 is an approximation to the 3rd Zemach moment $\langle r^3 \rangle_{(2)}$ depending on the charge form factor of the proton. Inserting the differing radii and their error bars given above

one gets for the deviation of the Lamb shift:

$$\delta(\Delta E_{\text{Lamb}}) = 0.322(46) \text{ meV} \quad (2)$$

where the error is dominated by the electronic experiments.

However, a recent non-perturbative relativistic calculation of the theoretical Lamb shift [6] yields somewhat different constants for the formula in eq. (1). From this follows $r_p = 0.83340(67) \text{ fm}$ resulting in a deviation $\delta(\Delta E_{\text{Lamb}}) = 0.403(51) \text{ meV}$. Considering the perfection of this calculation one is puzzled about possibilities of an explanation.

In view of the excellent accuracy of the muonic experiment and the very good agreement of the about half dozen electronic experiments, it is highly unlikely that the reason for the deviation is due to a problem on the experimental side. Therefore, one has to see what possibilities are left in the analysis of the data and in the framework of the QED description. At first one has to realize that the QED calculations of the muonic Lamb shift have been scrutinized again and again over the years. Of the many publications we mention some pertinent summaries [7–11]. On the other hand, there is also no room for a modification of the radiative corrections to the electron scattering cross sections.

In this paper it is shown that the wave functions determined by Carroll et al. [6] have been used incompletely as input in the QED description of the Lamb shift. If correctly used the radius puzzle disappears.

PREVIOUS PROPOSALS FOR EXPLAINING THE DIFFERENCE

Of the many proposals for explanations we list some prominent and typical.

- 3rd Zemach moment

One can construct an electric form factor with an evanescent charge cloud extending to very large radii but maintaining the radius of the electric experiments. This increases the 3rd Zemach moment $\langle r_p^3 \rangle_{(2)}$ so much that the muonic radius determined from eq.(1) agrees with the electric one [12–14]. However, it has been shown that such a conjecture is in disagreement with the measured form factors [15–18].

- Off-shell form factors

Miller et al. [19] have proposed off-shell influences of the form factors in the elastic box diagram contributing to the muonic hydrogen Lamb shift. This idea was refuted by Carlson and Vanderhaeghen who showed that such contributions are two orders of magnitude smaller than thought [20].

- Form factor extrapolation problems

The radius is derived from the electric and magnetic form factors $G_{E,M}$ according to $\langle r^2 \rangle^{1/2} = (6/G_{E,M}(Q^2))(dG_{E,M}(Q^2)/dQ^2)$ at exactly $Q^2 = 0$. Since in elastic electron scattering the form factor can be measured down to small but finite momentum transfers Q^2 only one might think about a change of the form factor at very small Q^2 . Carlson [21] refitted a subset of the data of Bernauer [22] at very low Q^2 but finally got results in agreement with the radius from the complete fits if higher order terms in the form factor Q^2 expansion were included. Wu and Kao [23] tried a "thorn" at very small Q^2 equivalent to increasing again the 3rd Zemach moment $\langle r_p^3 \rangle_{(2)}$ so much that agreement from eq.(1) results. However, here a dangerous oversight occurs since eq.(1) contains the first terms of the perturbative expansion series [24] only. Since the higher order moments diverge for such extreme modifications of the form factor at very small Q^2 eq.(1) this expansion breaks down and is not applicable anymore.

Recently the low Q^2 extrapolation has been formulated in the realm of chiral dynamics [25] putting it on firm theoretical grounds beyond the empirical Q^2 expansion. However, no change of the radius is indicated [26]. Similar ideas have been pursued also in ref.[27].

In a very recent paper Lorenz, Hammer, and Meißner [28] try their own fit to the data of Bernauer et al. [5, 22] using the "continued fraction" model for the electric and magnetic form factors $G_{E,M}$ [4]. However, of the many models tried by Bernauer et al. this model was one of the worst delivering unstable fits due to poles outside the fitted Q^2 range. Only for $Q^2 > 0.1 \text{ GeV}^2$ the fits of Lorenz, Hammer, and Meißner are sufficiently sta-

ble to allow an extrapolation to $Q^2 = 0$. However, the poles make such an extrapolation very questionable and consequently also the radius derived [29].

Also it has to be realized that the good agreement of the electronic Lamb shift and the electron scattering results make explanations with extrapolation modifications unlikely.

- Dispersion relations

In the already cited paper [28] Lorenz, Hammer, and Meißner present also a refit of their older dispersion relation fits complementing their old data base with the new data of Bernauer et al. [5]. As the old fits these new fits have a large $\chi_{red}^2 = 2.2$ for 100 degrees of freedom (dof) estimated from the figure; the dof are not given. Although the use of the theoretical χ^2 distribution is not really justified for these fits since the theory function is unknown (Bayesian situation) and the errors are not Gaussian distributed such a χ_{red}^2 is equivalent to a probability $P(\chi_{red}^2 > 2.2; dof \approx 100) \approx 10^{-10}$. The sensitivity of such fits to the radii is not discussed.

- Fancy particles beyond the standard model

In view of the convincing experimental evidence and the unsuccessful attempts with the mentioned conventional explanations some authors have speculated about new physics.

Batell, McKeen and Pospelov [30] consider new vector and scalar forces with carriers of less than 100 MeV mass. Tucker-Smith and Yavin [31] propose a new scalar or vector boson of about 1 MeV mass coupling more strongly to the muon than to the electron. Barger et al. [32] investigate the possibility of new particles with a special coupling to the muon more generally, but conclude that new spin-0, spin-1 and spin-2 particles are disfavored by other experimental constraints.

In a very recent paper Carlson and Rislow [33] discuss two models, one involving new particles with scalar and pseudoscalar couplings, and a second involving new particles with vector and axial couplings. Though it appears not impossible to accommodate the new particles, some fine tuning of masses and couplings is needed to adjust them to the used constraints of the Lamb shift, muon magnetic moments and kaon decay rate data.

AN OVERLOOKED ISSUE

The muonic Lamb shift is in leading order caused by the vacuum polarization in QED described by the exchange of an electron-positron loop. The loop correction can be calculated perturbatively with Feynman di-

agrams, see e.g. ref. [34, 35], or non perturbatively from the exact solution of the Dirac equation using Green's function [36, 37]. Not surprisingly both approximations give the same result in leading order of the expansion in $Z\alpha^2$. We stick to the first approach and note that it is equivalent to time dependent perturbation theory (tDEPpth). The energy shift is then given by the following expression:

$$E_{\text{Lamb},nl}^{(0)} = -\langle nl;0|\Sigma_{\mathcal{A}}^*|nl;0\rangle. \quad (3)$$

$|nl;0\rangle$ is the unperturbed solution of the wave equation of the muon in the external Coulomb field. $\Sigma_{\mathcal{A}}^*$ comprises in principle the self energy, the vertex correction and the vacuum polarization propagator, but for muonic hydrogen it suffices to restrict oneself to the exchange of one electron-positron loop, i.e. the vacuum polarization. The formula is derived using Feynman rules and a first order expansion in energy with respect to the unperturbed energy eigenvalue $E_{nl}^{(0)}$ ([34], chapter 14.2). Since the matrix element is of the form $\int \Sigma_{\mathcal{A}}^* \rho(r) r^2 dr$ with $\rho(r) = |nl;0|^2$ the charge distribution of the muon orbit, it is suggestive to define an external potential $V_{VP} = -\Sigma_{\mathcal{A}}^*$, the "Uehling potential". However, one has to be aware that this identification is not a trivial step but rather an approximation which has to be justified in the realm of QED. Weinberg shows how the external Coulomb potential emerges from the summation of multi photon exchange diagrams for a non relativistic two particle system ([34], chapter 13.6). A similar derivation had to be performed here in order to show the validity of the external potential approximation. In this approximation the vacuum polarization propagator is used as the external "Uehling potential" [38]. This point is essential here because it means one assumes that the muon can be regarded as a test charge in the "Uehling potential" [39]. The external "Uehling potential" is of order $Z\alpha^2$ and does not include higher order contributions as two loop exchanges, i.e. "double vacuum polarization" [35] or "vacuum polarization iteration" [9] or "polarization insertion in two Coulomb lines" Carroll et al. [6].

Since tDEPpth breaks down for systems with stationary bound states we have to solve a time independent wave equation ([34], chapter 14). This means one calculates the effect of the external "Uehling potential" V_{VP} on the bound state eigenvalues and wave functions by solving the wave equation:

$$H|nl\rangle = E_{nl}|nl\rangle \quad (4)$$

with

$$H = H_0 + V_{VP} = T + V_{\text{Coulomb}} + V_{VP} \quad (5)$$

where $|nl\rangle$ is now the new wave function and E_{nl} the new energy eigenvalue. If one solves eq.(4) numerically this eigenvalue is exact or "non-perturbative" [6] in the realm

of time independent perturbation theory (tINDEPpth), but it stays first order in tDEPpth. Consequently, we shall call the energies and wave functions calculated with V_{VP} "perturbed" and without "unperturbed". We can easily bring eq.(4) in a form corresponding to eq.(3). We set $|nl\rangle = |nl;0\rangle + \delta|nl\rangle$ with $\langle nl;0|nl;0\rangle = 1$ and $\delta|nl\rangle \perp |nl;0\rangle$. Multiplying eq.(4) with $\langle nl;0|$ from the left we get:

$$E_{nl} - E_{nl}^{(0)} = \langle nl;0|V_{VP}|nl\rangle \quad (6)$$

The normalization choice for $|nl;0\rangle$ and $|nl\rangle$ is the same as in conventional tINDEPpth yielding this simple form and making the situation lucid. However, one has to realize that the perturbed state $|nl\rangle$ is not normalized to 1. As long as one does not make the relation to the unperturbed state explicit this means just an overall factor leaving the energy eigenvalue unchanged. But here we do relate to $E_{nl}^{(0)}$ and one has to consider this carefully. (For a discussion of this point in the frame work of tINDEPpth (see [40], chapter 5.1.)

We emphasize that the perturbed, i.e. exact, solution contains only contributions of order $Z\alpha$ (Coulomb potential) and $Z\alpha^2$ ("Uehling potential"). This is evident if one looks at the expansion of these states in tINDEPpth [40]. The expansion coefficients are the matrix elements of the perturbation V_{VP} and mix the discrete unperturbed eigenstates. The idea that the electron-positron loop is reiterated in these solutions as in the Lippmann-Schwinger equation, i.e. tDEPpth, is not correct for bound states [41], chapter 6.3. Stationary bound states cannot be reached in such an expansion since they are eigenstates with discrete eigenvalues. The point is that vacuum polarization loops occur only in QED and can be approximated in a wave equation only by an external potential, i.e. the "Uehling potential" here. Therefore, it is not correct to assume that the "polarization insertion in two Coulomb lines", Carroll et al. [6], is already present in the solution of the time independent wave equation eq. (6). Rather this contribution has to be calculated separately applying Feynman rules and added to the propagator $\Sigma_{\mathcal{A}}^*$.

From this discussion follows that eq. (6) does not represent a "non-perturbative" Lamb shift. In order to calculate the corrected Lamb shift with the perturbed wave function one has to propagate these wave functions with the propagator $\Sigma_{\mathcal{A}}^*$. In other words, one has to insert the approximate wave function gained from tINDEPpth with the external "Uehling potential" in the tDEPpth. One could think that this reinsertion is a double application of the Uehling effect, however, it rather realizes the dynamic effect on the energy shift due to the vacuum polarization through the perturbed wave function. The forward-backward change between the "Uehling potential" as a propagator and an external potential is, of course, somewhat intuitive but we have no better approximation. The procedure resembles in a way the self

consistent construction of the orbits of a multi electron atom.

The derivation of this procedure is implied in the derivation of the Lamb shift in Weinberg's book ([34], chapter 14.2). Instead of expanding the perturbed solutions of the wave equation $U_N = u_N + \delta u_N$, or $|n l\rangle = |n l; 0\rangle + \delta |n l\rangle$ in the notation used here, with u_N the unperturbed wave function and neglecting the terms with δu_N one keeps the U_N . Since the Uehling effect relates to the unperturbed eigenvalue $E_{nl}^{(0)}$ one has again to expand the energy denominator with respect to this energy. The result is a corrected formula for the Lamb shift $E_{\text{Lamb}, nl}$ where one replaces the normalized unperturbed wave function $|n l; 0\rangle$ by the normalized perturbed $|n l\rangle_N$ in eq.(3):

$$E_{\text{Lamb}, nl} = {}_N \langle n l | V_{VP} | n l \rangle_N \quad (7)$$

This expression is intuitively plausible and has been already mentioned by Carroll et al. [6], but, was not used and discussed by them.

We can interpret eq.(7) as the propagation of the perturbed wave function $|n l\rangle$ by the vacuum polarization propagator, where, of course, the initial and final states have to be identical. If we compare eq.(6) to eq.(7) we realize that the perturbed state $\langle n l |$ appears only once in eq.(6). This means that one neglects the dynamical feed back effect of the perturbed wave function on the vacuum polarization, i.e. the additional energy shift caused by it.

Carroll et al. [6] take the difference $E_{2P} - E_{2S}$ as the corrected Lamb shift with E_{2P} and E_{2S} the "non-perturbative" energy eigenvalues calculated by integrating the Dirac equation numerically. At first glance one could believe that this is correct since the degenerate eigenvalues $E^{(0)} = E_{2S}^{(0)} = E_{2P}^{(0)}$ cancel, but by comparing eq.(6) and eq.(7) we see that the dynamic feed back effect is missing. We shall see in the next section that this explains the difference between the calculation of Carroll et al. [6] and the evaluation of the Lamb shift with the perturbed wave functions.

NUMERICAL REALIZATION

In distinction to Carroll et al. in ref. [6] we restrict ourselves to the non relativistic Schrödinger equation with the Coulomb potential of a point charge. This suffices for demonstrating the correction of the Lamb shift without considering fine structure splitting, finite size effect, etc.. Relativistic effects on the Lamb shift are small [9] and do not change anything essential for this discussion. This means we calculate the exact solutions of the Schrödinger equation with the point Coulomb potential and the external "Uehling potential" added according to eq.(5).

A compact representation of the "Uehling potential" V_{VP} , well suited for our calculation, is the representation

provided by Pachucki [35]:

$$V_{VP}(r) = -\frac{Z\alpha}{r} \frac{\alpha}{\pi} \int_4^\infty \frac{d(q^2)}{q^2} \exp(-m_e q r) u(q^2) \quad (8)$$

where

$$u(q^2) = \frac{1}{3} \sqrt{\left(1 - \frac{4}{q^2}\right)} \left(1 + \frac{2}{q^2}\right) \quad (9)$$

and q^2 is the internal momentum squared normalized to m_e^2 . For the radial unperturbed wave functions we use:

$$R_{2S} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{a_0^3}} \exp\left(-\frac{r}{2a_0}\right) \left(1 - \frac{r}{2a_0}\right) \quad (10)$$

$$R_{2P} = \frac{1}{2\sqrt{6}} \frac{1}{\sqrt{a_0^3}} \exp\left(-\frac{r}{2a_0}\right) \frac{r}{a_0} \quad (11)$$

with $a_0 = \hbar c / (\alpha \mu)$ the Bohr radius and μ the reduced mass. Using these wave functions in eq.(3) we get for the leading order Lamb shift $\Delta E_{\text{Lamb}} = E_{\text{Lamb}, 2P}^{(0)} - E_{\text{Lamb}, 2S}^{(0)} = 205.005 \text{ meV}$ in agreement with [35].

For the numerical integration of the Schrödinger equation we have used Mathematica. As Carroll et al. [6] we have made extensive tests to guaranty the quality of the solutions. All calculations have been made with an internal precision of 64 digits and an accuracy goal of 20 digits. The optimal method is the "Explicit Runge-Kutta" integration for the S-State and the change between various methods provided "automatically" for the P-state. The numerical eigenvalues of the unperturbed $2S$ and $2P$ states, i.e. without the "Uehling potential", are compared to the non relativistic exact solution (Bohr energies) and found to be good to a few neVs for different boundary conditions at small ($\approx 0.1 \text{ fm}$) and large ($\approx 10000 \text{ fm}$) radii to which the eigenvalues are sensitive. Of course, since we do not take the difference of the large energy eigenvalues as Carroll et al. [6] but calculate the small Lamb shift only, we do not really need this extreme accuracy. However, when using our unperturbed numerical eigenvalues we get $\Delta E_{\text{Lamb}} = 205.005 \text{ meV}$ in complete agreement with the calculation using exact wave functions of eq.(10) and eq.(11). The eigenvalues with the "Uehling potential" have been determined using the virial theorem.

Figure 1 and Fig. 2 show the difference of the density of the unperturbed state $|n l; 0\rangle^2 r^2$ minus the density of the perturbed state $|n l\rangle_N^2 r^2$.

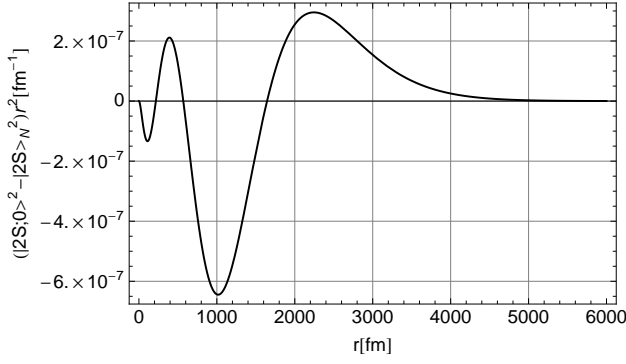


FIG. 1. The difference $(|2S;0\rangle^2 - |2S\rangle_N^2)r^2$ showing the polarization charge density divided by the negative elementary charge of the muon due to the "Uehling potential" for the $2S$ state

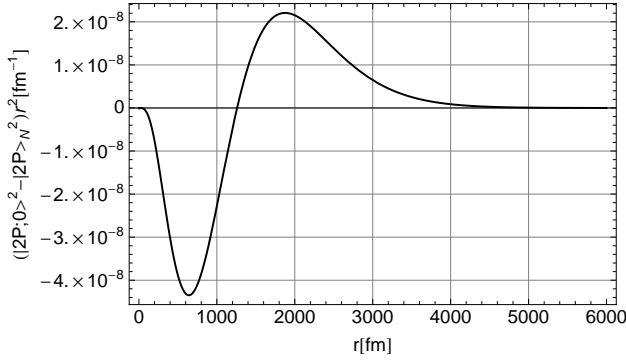


FIG. 2. The difference $(|2P;0\rangle^2 - |2P\rangle_N^2)r^2$ showing the polarization charge density divided by the negative elementary charge of the muon due to the "Uehling potential" for the $2P$ state

One nicely sees the polarization charge induced in the vacuum by the muon. The wiggles at small radii in Fig. 1 are no numerical artifacts but due to the double bump structure of the $2S$ state. In order to get an idea of the scales we note that the Bohr radius for the muon is $a_0 = 285$ fm, the scale of the Compton wave length of the electron or positron in the electron-positron pair of the vacuum polarization $\lambda = 2426$ fm, the rms radius of the $2S$ state $\langle 2S|r^2|2S\rangle^{1/2} = 1854$ fm, and for the $2P$ state $\langle 2P|r^2|2P\rangle^{1/2} = 1560$ fm. As expected some positive charge is pushed to larger radii compensated by negative charge at small radii indicating the induction of the polarization cloud in the vacuum. An analytical derivation of this polarization is derived in ref. [34], chapter 11.2.

Calculating the difference of the Lamb shifts with the normalized perturbed $2S$ and $2P$ states according to eq.(7) one gets the salient result of this paper:

$$\Delta E_{\text{Lamb}}^{\text{point charge}} = 205.307(1)\text{meV} \quad (12)$$

where the error is a best estimate from the variation of

the value with different integration boundaries. Comparing this result to the canonical value for the unperturbed wave functions $\Delta E_{\text{Lamb}}^{(0),\text{point charge}} = 205.005(1)$ one arrives at

$$\delta(\Delta E_{\text{Lamb}}) = 0.302(1)\text{meV} \quad (13)$$

in very good agreement with the searched for difference of eq. (2).

For the difference of the eigenvalues from the numerical solution of the wave equation according to eq.(6) we get $E_{2P} - E_{2S} = 205.156(1)\text{meV}$. If we take directly the eigenvalues $E_{2P} - E_{2S}$ from the numerical solution of the wave equation eq.(4) as Carroll et al. [6] we get $205.159(3)\text{meV}$. The deviation from the difference calculated with eq.(6) indicates a limit of the numerical accuracy. Since we do not use the difference of the eigenvalues we have not insisted to improve this limit. Considering the relativistic correction due to the Dirac wave functions of 0.021meV [35], missing in our non relativistic calculation, this is in good agreement with the relativistic result of Carroll et al. [6] of $205.1706(5)\text{meV}$.

As already discussed this result does not include the higher order $Z\alpha^4$ contribution as the two loop vacuum polarization exchange. Therefore, the "double vacuum polarization" contribution of 0.151meV [35] has to be added to our result $\delta(\Delta E_{\text{Lamb}})$ in eq. (13) as well as to that of Carroll et al. [6] of $205.1706(5)\text{meV}$, who, however, neglected it.

CONCLUSIONS

If the Lamb shift is taken as the dynamical QED effect due to the interaction of the muon with the vacuum polarization and not as a shift caused by the "Uehling potential" as an static external potential, one gets agreement for the radius determined from the Lamb shift in muonic hydrogen with the combined electronic experiments. Since the relativistic calculations including the finite size effects of Carroll et al. [6] have to be redone realizing the new considerations of this paper we stick to the formula eq.(1) used by Pohl et al. [1]. This means all other corrections, in particular the "double vacuum polarization" and the finite size effect, are the same as used in that analysis. If we correct the Lamb shift of the point charge in eq.(1) to the value calculated in this paper, we arrive at a new value for the rms radius of the proton derived from the muon experiment:

$$\langle r^2 \rangle^{1/2} = 0.87650(71)\text{fm} \quad (14)$$

where we have taken the recent 3rd Zemach moment from ref. [16] and included the estimated error of the point charge Lamb shift. This value is now in good agreement with the best electron scattering rms radius $0.879(8)\text{fm}$ [5].

Again QED wins and there is no reason to fear a "chink in the armor" as in ref. [42]. However, it is also true that "it must be repeated that the theory of relativistic effects and radiative corrections in bound states is not yet in entirely satisfactory shape" [34], p. 560.

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